UNIVERSITY OF PENNSYLVANIA

ON TWO-DIMENSIONAL ANALYSIS SITUS WITH SPECIAL REFERENCE TO THE JORDAN CURVE-THEOREM

DUDLEY WELDON WOODARD

A THESIS

IN MATHEMATICS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

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On two-dimensional analysis situs with special reference to the Jordan curve-theorem.

By

D. W. Woodard (Philadelphia).

Introduction. R. L. Moore, in his paper, „*On the Foundations of Plane Analysis Situs*", Transactions of the American Mathematical Society, Volume 17, 1916, proposed three systems of axioms, Σ_1, Σ_2, Σ_3, for the development of two-dimensional analysis situs. This paper will hereafter be referred to as „Moore". To facilitate reference, Moore's notation will be followed as closely as practicable.

In Axiom 8, which belongs to all three systems, Moore assumes that every simple closed curve is the boundary of a region, that is, that every simple closed curve defines a bounded connected set of connected exterior having further properties implied by certain other axioms of the three systems.

The chief purpose of this investigation is to replace Moore's Axiom 8 by another axiom of such nature that no property of the simple closed curve is assumed. The Jordan curve-theorem in its most general form appears as the fundamental theorem of the set of theorems. Two systems of axioms, I and II, are presented. It is proved that (1) all of Moore's theorems follow as consequences of each of the systems of axioms, (2) every simple closed curve is the boundary of a set of points having all the properties of a region, (3) every space satisfying the set designated as Axioms I is homeomorphic with the Euclidean plane and (4) there exist spaces satisfying Axioms II that are „neither metrical, descriptive, nor separable" [1].

[1]) Axioms II are satisfied by the space thus described by Moore, page 164.

The proof of the Jordan curve-theorem for spaces of the type described in (4) constitutes, perhaps, the most interesting result in the development that follows.

If Axiom 6 is omitted, it is still possible to prove Theorems 1—5 upon the basis of the remaining axioms, except that, in the case of Theorem 1, the last six words must be changed to read „H_i' is a subset of H_{i-1}'". The facts developed in the proofs of Theorems 1—5 constitute a proof of the Jordan curve-theorem. It is of interest to note that, under these circumstances, the proof of the Jordan curve-theorem is based upon a set of axioms which contains no assumption as to the character of the boundaries of regions. The foregoing statements apply to either of the two sets of axioms.

For an account of related investigations previously published, one is referred to the paper by J. R. Kline on *„Separation Theorems and their Relation to Recent Developments in Analysis Situs"* [1]).

I wish to express my deep obligation to Dr. J. R. Kline who suggested the problem and whose helpful criticism has been of inestimable value.

Definitions. Let S denote the set of all elements (points) to be considered. Regions are certain subsets of S having the properties implied in the axioms. For definitions of „limit point of a point-set", „boundary of a point-set", „connected set", „Heine-Borel property", „simple continuous arc", „simple closed curve", „bounded set", see Moore, pages 132, 135, 136, 139. The set of points composed of the point-set M and its boundary is denoted by M'. The expressions, „simple continuous arc ABC", „arc ABC" and „ABC", denote the same object. Frequently an arc is designated by naming its two end-points, as, AC. The symbol $\underline{ABC}$ denotes the point-set (arc ABC) $- A - C$.

Axioms I. Axioms 1—5 are given precisely as they occur in Moore.

[1]) In this connection, attention is called to the Heidelberg thesis of Imgard Gawehn: *„Über unberandete 2-dimensionalen Mannigfaltigkeiten"*, Math. Ann. 98, 1927. In this thesis the validity of the Jordan curve-theorem im kleinen is postulated. In a reference to Moore's paper, the following incorrect statement is made: „ausserdem wird viel vorausgesetzt, z. B. die Existenz spezieller Systeme einander nicht schneidende Jordankurven".

Axiom 1. *There exists an infinite sequence of regions* $K_1, K_2, K_3, \ldots$, *such that* (1) *if* m *is an integer and* P *is a point, there exists an integer* n, *greater than* m, *such that* K_n *contains* P, (2) *if* P *and* $\overline{P}$ *are distinct points of a region* R, *then there exists an integer* δ *such that if* $n > \delta$ *and* K_n *contains* P, *then* K_n' *is a subset of* $R - \overline{P}$.

Axiom 2. *Every region is a connected set of points.*

Axiom 3. *If* R *is a region,* $S - R'$ *is a connected set of points.*

Axiom 4. *If* R *is a region,* R' *possesses the Heine-Borel property.*

Axiom 5. *There exists an infinite set of points that has no limit point.*

Axiom 6. *If* P *is a point on the boundary of a region* R, *then there exist two simple continuous arcs* AP *and* BP *such that* $AP - P$ *and* $BP - P$ *are subsets of* R *and* $S - R'$ *respectively.*

Axiom 7. *If* AB *is a simple continuous arc lying within a region* R, *then* $R - AB$ *is a connected set of points.*

Axiom 8. *If* α_1 *and* α_2 *are two finite connected sets of regions whose sum is a connected set, and* R_1 *and* R_2 *are two mutually exclusive regions without common boundary points such that* R_i $(i = 1, 2)$ *contains at least one point common to* α_1 *and* α_2, *and all points common to* $(\alpha_1)'$ *and* $(\alpha_2)'$ *belong to* $R_1' + R_2'$, *then there exists at least one region* H *containing* $\alpha_1 + \alpha_2$ *such that* (1) *the boundary of* H *is a subset of the boundary of* $\alpha_1 + \alpha_2$ *and* (2) H *contains a connected subset* M *exterior to* $R_1 + R_2$ *and containing at least one point of* α_1 *and one point of* α_2.

Independence examples. The independence examples given by Moore, page 162, for Axioms 1—5, 8 of his Σ_1 system will serve for the above axioms bearing the same number. In the case of Axiom 6, Moore's E_6 or E_7 may be used to obtain a space with regions so defined that Axioms 1—5, 7, 8 hold true but for which one or the other of the arcs indicated in Axiom 6 does not exist. Certain types of regions, in addition to those indicated by Moore, must be defined in order that Axiom 8 may be satisfied in the independence examples for Axioms 4 and 6. The needed definitions are obvious.

As an independence example for Axiom 7, consider the set of points S_7 contained in the ordinary straight line α and in β, the half-plane determined by α in a given Euclidean plane. A set of points is a region R if and only if (1) R is an ordinary Jordan region such that R' is a subset of the half-plane β, or R is the

finite part of S_7 determined by an ordinary arc ACB such that A and B belong to α and ACB is a subset of β. All axioms are satisfied except Axiom 7.

Preliminary Theorems. Theorems 1—22, Moore, which depend upon Axioms 1—6 are assumed. The following property of the simple continuous arc as stated by Moore, page 139, is of particular importance in our work: If an arc AB has at least one point in common with a closed set F, then AB has a first and a last point in common with F.

Theorem 1. *If J is a closed curve, there is a sequence of pairs of region-sets $\{\alpha_1^{(i)}, \alpha_2^{(i)}\}$, $(i = 1, 2, 3, \ldots)$ such that* (1) *J is the common part of $\alpha_1^{(1)} + \alpha_2^{(1)}$, $\alpha_1^{(2)} + \alpha_2^{(2)}, \ldots$ and* (2) *there exists a corresponding sequence of regions $\{H_i\}$ such that the boundary of H_i is a subset of the boundary of $\alpha_1^{(i)} + \alpha_2^{(i)}$, H_i contains $\alpha_1^{(i)} + \alpha_2^{(i)}$ and H_i' is a subset of H_{i-1}.*

Proof. There exist points A and B separating on J the points C and D. All regions used are members of the fundamental sequence of regions postulated by Axiom 1. Let R_A and R_B be two regions about A and B respectively such that R_A' and R_B' have no common points, and R_A' and R_B' contain no points of the arcs CBD and CAD respectively. Cover each point of ACB not in $R_A + R_B$ by a region having no point or boundary point in common with ADB. These regions, together with $R_A + R_B$, cover ACB. A finite subset of these regions, including R_A and R_B, covers ACB (Moore, Theorem 12). Let $\alpha_1^{(1)}$ denote this finite set of regions. Cover each point of ADB not in $R_A + R_B$ by a region having no point or boundary point in common with ACB or with any region of the set $\alpha_1^{(1)} - R_A - R_B$. As before, we obtain a finite set of regions, including R_A and R_B, which cover ADB. Let $\alpha_2^{(1)}$ denote this finite set of regions. Evidently $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ satisfy the requirements of Axiom 8 and the corresponding region H_1 exists. H_1 is unique. This follows immediately from Theorem 21, Moore. In obtaining $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$, all regions used have subscripts greater than 2, and, in general, in obtaining $\alpha_1^{(i)}$, $\alpha_2^{(i)}$, all regions used have subscripts greater than i. There exist regions $R_1^{(2)}$ and $R_2^{(2)}$ about A and B respectively such that $R_1^{(2)}$ and $R_2^{(2)}$, together with their boundaries, are subsets of R_A and R_B respectively. As in the preceding, there exists a finite set of regions covering all points of ACB not in $R_1^{(2)} + R_2^{(2)}$ such that every region of the set with its boundary is a subset of $\alpha_1^{(1)}$ and no region of the set has a point or boundary point in common with

ADB. There is again a finite set of regions covering ACB and including $R_1^{(2)}$ and $R_2^{(2)}$. We can obtain similarly a finite set of regions, including $R_1^{(2)}$ and $R_2^{(2)}$, covering ADB such that each region of the set with its boundary is a subset of $\alpha_2^{(1)}$ and no region of the set other than $R_1^{(2)}$ and $R_2^{(2)}$ contains a point of ACB. Let $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ denote the first and second of these finite sets respectively. The corresponding region H_2 exists. Continuing in this manner, we can obtain the required sequence of pairs of region-sets and the corresponding sequence of regions $\{H_i\}$. By Moore, Theorem 21, H_i' is a subset of H_{i-1}.

Let Q be a point not belonging to J. Since J is closed, there exists a region R_Q about Q such that R_Q' contains no point of J. Let Q_1 be a point of R_Q distinct from Q. By Axiom 1, there is a number δ such that for $n > \delta$, K_n' is a subset of $R_Q - Q_1$, provided K_n contains Q. In the process of obtaining the sequence of pairs of region-sets, every region of the fundamental sequence belonging to $\alpha_1^{(i)} + \alpha_2^{(i)}$ had a subscript greater than i. Then, no region of $\alpha_1^{(\delta+1)} + \alpha_2^{(\delta+1)}$ can contain Q. Hence Q cannot belong to the common part of $\alpha_1^{(1)} + \alpha_2^{(1)}$, $\alpha_1^{(2)} + \alpha_2^{(2)}, \ldots$, and this common part is identical with J.

Theorem 2. *Corresponding to every simple closed curve J, there is a unique bounded set I_J such that I_J and E_J, the complement of $I_J + J$, are separated by J.*

Proof. The regions H_1, H_2, $H_3, \ldots$, obtained in the last theorem, have a common part H_ω which is closed (Moore, Theorem 14). Denote $H_\omega - J$ by I_J. Assume that I_J is non-vacuous and contains a point Q. The point Q cannot be a limit point of E_J. For, if Q were a limit point of E_J, every region R_Q containing Q would contain a point Q_1 of E_J. Then, Q_1 belongs to $S - H_i'$ for some value of i. An arc in R_Q joining Q to Q_1 [1]) would contain a point of the boundary of H_i. Under the assumption, then, Q is a limit point of the sum of the boundaries of the regions H_i. By the preceding theorem, Q does not belong to the common part of $\alpha_1^{(1)} + \alpha_2^{(1)}$, $\alpha_1^{(2)} + \alpha_2^{(2)}, \ldots$. Then, there is a number n such that Q does not belong to $(\alpha_1^{(n)} + \alpha_2^{(n)})'$. There is a region $\overline{R}_Q$ containing Q and no point of $(\alpha_1^{(n)} + \alpha_2^{(n)})'$. The boundary of H_{n+j} $(j \geqq 1)$ is a subset of $\alpha_1^{(n)} + \alpha_2^{(n)}$. Then $\overline{R}_Q$ contains no point of the boundary of H_{n+j}.

[1]) Moore, Theorem 16.

Hence Q is not a limit point of E_J. Further, in the same manner we may prove that no point of E_J is a limit point of I_J.

The regions $R_1^{(i)}$ and $R_2^{(i)}$, containing all points common to $\alpha_1^{(i)}$ and $\alpha_2^{(i)}$, belong to R_A and R_B respectively. Then, by Axiom 8, there exists in H_i a connected set M_i exterior to $R_A + R_B$ such that M_i contains a point $A_1^{(i)}$ of $\alpha_2^{(i)}$ and a point $A_2^{(i)}$ of $\alpha_2^{(i)}$. But $A_1^{(i)}$ and $A_2^{(i)}$ belong to $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ respectively. With due regard to the fact that M_i is connected and that $\alpha_1^{(1)} - R_A - R_B$ and $\alpha_2^{(1)} - R_A - R_B$ have no common point or boundary point, it follows easily that H_i contains a point F_i exterior to $\alpha_1^{(1)} + \alpha_2^{(1)}$. We thus obtain a sequence of points $F_2, F_3, \ldots, F_n, \ldots$, such that F_n belongs to H_n and is exterior to $\alpha_1^{(1)} + \alpha_2^{(1)}$. Since each point belongs to H_1, the sequence is bounded and possesses a limit point F (Moore, Theorem 13). The point F obviously cannot belong to $\alpha_1^{(1)} + \alpha_2^{(1)}$ or to E_J or to J. In any event, the point F belongs to I_J. Then, I_J is non-vacuous. It can easily be proved that E_J is connected [1]).

Suppose that I_J is not unique, that is, suppose that for two different processes of the kind described above, there were obtained two sets H_ω and $\overline{H}_\omega$ such that $\overline{H}_\omega$ contains a point P not in H_ω. The set $S - H_\omega$ is connected. Let E be a point of $S - H_\omega$ and $S - \overline{H}_\omega$. Then, under the assumption made above, there exists an arc PE in $S - H_\omega$ which contains a point P_1 on the boundary of $\overline{H}_\omega$. As proved above, no point of $\overline{H}_\omega - J$ is a boundary point of $\overline{H}_\omega$. Then, P_1 belongs to J. But J belongs to H_ω. This contradiction shows that H_ω, and therefore I_J, is unique. The set I_J, being a subset of H_1, is bounded.

Theorem 3. *Every point of a simple closed curve J is a limit point of I_J.*

Proof. Assume that P, any point of J, is not a limit point of I_J. An arc joining a point of I_J to a point of E_J [2]) contains a first point on J. Then J contains a non-vacuous subset N consisting of points which are limit points of I_J. The set N is closed. Then P is not a limit point of N. There exists a region R_P about P such that R_P' contains no point of N. There is in R_P an arc APB of J. The arc APB contains no point of N. Let C be a point of J such

[1]) In fact, it can be shown that every two points of E_J are the extremities of an arc lying wholly in E_J.

[2]) Moore, Theorem 15, 22.

that P and C separate A and B on J. Let D_1 and D_2 respectively be the first points that PAC and PBC have in common with the closed set N. The set $D_1 C D_2 + I_J$ is closed and connected.

Form $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ in accordance with Axiom 8 as follows: $R_1^{(1)}$ and $R_2^{(1)}$ are regions about D_1 and D_2 respectively such that $R_1^{(1)}$ and $R_2^{(1)}$ have no point or boundary point in common. The set $\alpha_1^{(1)}$ consists of $R_1^{(1)} + R_2^{(1)}$ and a finite set of regions covering all points of $D_1 P D_2$ not in $R_1^{(1)} + R_2^{(1)}$ such that no region of $\alpha_1^{(1)} - R_1^{(1)} - R_2^{(1)}$ has a point or boundary point belonging to $D_1 C D_2 + I_J$. The set $\alpha_2^{(1)}$ consists of $R_1^{(1)} + R_2^{(1)}$ and a finite set of regions covering all points of $D_1 C D_2 + I_J$ not in $R_1^{(1)} + R_2^{(1)}$ such that no region of $\alpha_2^{(1)} - R_1^{(1)} - R_2^{(1)}$ has a point or boundary point in common with a region of $\alpha_1^{(1)} - R_1^{(1)} - R_2^{(1)}$. Evidently regions of the kind described exist so that $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ can be obtained so as to satisfy the requirements of Axiom 8. Let $R_1^{(2)}$ and $R_2^{(2)}$ be regions which, with their boundaries, lie in $R_1^{(1)}$ and $R_2^{(1)}$ respectively and contain the points D_1 and D_2 respectively. The set $\alpha_1^{(2)} - R_1^{(2)} - R_2^{(2)}$ is a finite set of regions which cover all points of $D_1 P D_2$ not in $R_1^{(2)} + R_2^{(2)}$ and which, with their boundaries, are subsets of $\alpha_1^{(1)}$. The set $\alpha_2^{(2)} - R_1^{(2)} - R_2^{(2)}$ is a finite set of regions which cover all points of $D_1 C D_2$ not in $R_1^{(2)} + R_2^{(2)}$ and which, with their boundaries are subsets of $\alpha_1^{(1)}$. The sets $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ evidently exist in satisfaction of the requirements of Axiom 8. From this point proceed precisely as in Theorem 1 and obtain the corresponding sequence of regions, $H_1, H_2, H_3, \ldots$, whose common part is $I_J + J$. By the preceding theorem, I_J contains a point F not in $\alpha_1^{(1)} + \alpha_2^{(1)}$. But $\alpha_1^{(1)} + \alpha_2^{(1)}$ contains I_J. Thus, we are led to a contradiction by the assumption that any point P of J is not a limit point of I_J.

Corollary. *No region is a subset of an arc.*

This follows easily by the proof given by Moore of Theorem 23. The interior of a simple closed curve is to be interpreted to be an I_J-set of points.

Theorem 4. *Every point of a simple closed curve J is a limit point of E_J.*

Proof. Let P be any point of J and R_P any region containing P. Then R_P contains an arc APB of J. The curve J contains at least one point Q which is a limit point of E_J. By Theorem 1, there is a region H containing J. There is a region R_Q containing Q such that R_Q lies in H and contains a point E of E_J. By Axiom 7,

there exists in H an arc EP containing no point of arc $(J-APB)$ such that P_1 is the first point that EP has on APB. P_1 is a limit point of E_J and lies in R_P. Then any region containing P contains a point of J which is a limit point of E_J. Hence P is a limit point of the subset of J consisting of limit points of E_J. Then, P is a limit point of E_J.

Theorem 5. *The set I_J defined by any simple closed curve J is connected.*

Proof. Let Q be a point of I_J and D_Q, the maximal connected subset of I_J containing Q. There is an arc QE joining Q to a point E of E_J having D_1 as the first point after Q on J. Let R_{D_1} be a region containing D_1 such that $Q+E$ does not belong to R'_{D_1}. There exists an arc QE in $S-R'_{D_1}$ having D_2 as the first point after Q on J. Then D_1 and D_2 are limit points of D_Q. Let N be the subset of J containing all points of J which are limit points of D_Q. Suppose that a point P of J is not a limit point of D_Q, and let C be a point on J such that P and C separate D_1 and D_2 on J. As in Theorem 3, there exist two points A and B on J such that A and B are limit points of D_Q and the arc ACB contains N. There is an arc XYZ such that $\underline{XYZ}$ belongs to D_Q and X and Z are points on J in R_A and R_B, regions about A and B respectively, such that $R'_A+R'_B$ does not contain P, and R'_A and R'_B have no common points. Under the assumption that P is not a limit point of D_Q, no point of APB not in R_A+R_B is a limit point of D_Q. Let T_1 and V_1 be points of R_A, and T_2 and V_2 points of R_B such that these points lie on J_1 (the simple closed curve $PXYZP$) in the order $PT_1XV_1YV_2ZT_2$, and T_1XV_1 and T_2ZV_2 are subsets of R_A and R_B respectively. $\underline{XCZ}$ belongs to E_{J_1} (Theorem 4). Let S_1 denote the set of points of I_{J_1} not in R_A+R_B. Then each point of S_1 belongs to one and only one of the following classes.

(1) A point P_1 of S_1 belongs to C_1 if there exists an arc lying in I_{J_1} except for one end-point and joining P_1 to a point of T_1PT_2 and a similar arc joining P_1 to a point of V_1YV_2.

(2) A point P_2 of S_1 belongs to C_2 if there exists an arc lying in I_{J_1} except for one end-point and joining P_2 to a point of T_1PT_2, but there is no similar arc joining P_2 to a point of V_1YV_2.

(3) A point P_3 of S_1 belongs to C_3 if there exists an arc lying in I_{J_1} except for one end-point and joining P_3 to a point of V_1YV_2, but there is no similar arc joining P_3 to a point of T_1PT_2.

(4) All other points of S_1 belong to C_4.

No point of C_i is a limit point of C_j $(i \neq j)$. It is to be noted that the boundary points in I_{J_1} of R_A and of R_B belong to one of these four classes.

Suppose that C_1 is vacuous. Form $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ of Axiom 8 as follows: Let R_A and R_B be $R_1^{(1)}$ and $R_2^{(1)}$ respectively. (In this case, and in several instances which follow, $R_1^{(i)}$ and $R_2^{(j)}$ are common to $\alpha_1^{(i)}$ and $\alpha_2^{(j)}$). Since every point of I_{J_1} can be joined to some point of J by an arc lying in I_{J_1} except for one end-point, every point of C_4 can be joined to some point on the boundary of R_A or R_B by an arc lying in I_{J_1}. C_4 may be vacuous. All regions used in the following are regions of the fundamental seqence of regions of Axiom 1. Cover each point of T_1PT_2 not in $R_A + R_B$ with a region having no point or boundary point in common with $C_3 + XYZ$. Cover each point of $C_2 + C_4$ with a region that lies in I_{J_1} and has no point or boundary point in common with C_3. Then, these regions, together with R_A and R_B, cover the closed set $XPZ + C_2 + C_4$. A finite subset, including $R_A + R_B$ will cover the same set of points. Denote this finite set of regions by $\alpha_1^{(1)}$. The set $\alpha_1^{(1)}$ is connected. Cover each point of XYZ not in $R_A + R_B$ and each point of C_3 with a region having no point or boundary point in common with XPZ or with any region of $\alpha_1^{(1)} - R_A - R_B$. A finite subset of these regions, together with $R_A + R_B$, will cover the closed set $C_3 + XYZ$. Denote this finite set, which includes $R_A + R_B$, by $\alpha_2^{(1)}$. Keeping in mind that R_A and R_B take the place of $R_1^{(1)}$ and $R_2^{(1)}$ respectively, it is evident that $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ satisfy the requirements of Axiom 8. Form $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ as follows, using only regions of subscript greater than 2. The regions $R_1^{(2)}$ and $R_2^{(2)}$ are regions about X and Z respectively such that $R_1^{(2)}$ and $R_2^{(2)}$, with their boundaries, are subsets of R_A and R_B respectively. $\alpha_1^{(2)} - R_1^{(2)} - R_2^{(2)}$ consists of a finite number of suitably chosen regions covering points of XPZ not in $R_1^{(2)} + R_2^{(2)}$ and such that these regions with their boundaries lie in $\alpha_1^{(1)}$. The set $\alpha_2^{(2)} - R_1^{(2)} - R_2^{(2)}$ consists of a finite set of regions covering points of XYZ not in $R_1^{(2)} + R_2^{(2)}$. These regions evidently exist in such wise that $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ satisfy the requirements of Axiom 8. From this point proceed as in Theorem 1 and obtain the sequence of regions, $H_1, H_2, H_3, \ldots$ whose common part is $I_{J_1} + J_1$. But, as proved previously, I_{J_1} contains a point F not in $\alpha_1^{(1)} + \alpha_2^{(1)}$. But $\alpha_1^{(1)} + \alpha_2^{(1)}$ contained I_{J_1}. This contradiction shows that C_1 is not

vacuous. Then, there is an arc [1]) in I_{J_1} joining a point of $T_1 P T_2$ to a point of $V_1 Y V_2$. (I_{J_1} is obviously a subset of I_J). This contradicts the fact that APB contains no point which is a limit point of D_Q. Hence P is a limit point of D_Q. Then, every point of J is a limit point of D_Q.

Let P and Q be any two points of I_J. There exist D_P and D_Q as defined above. Suppose that there is no arc in I_J joining a point of D_P to a point of D_Q. Since every point of J is a limit point of D_P and of D_Q, there are two arcs, $P_1 P_2 P_3$ and $Q_1 Q_2 Q_3$, such that P_1, P_3, Q_1, Q_3 are distinct and lie on J, $\underline{P_1 P_2 P_3}$ and $\underline{Q_1 Q_2 Q_3}$ belong to D_P and D_Q respectively and, under the assumption, have no point in common. The points P_1, P_3, Q_1, Q_3, together with two other points A and B, may obviously be so chosen that they lie on J in the order $A P_1 Q_1 B Q_3 P_3$. Let J_1 denote the closed curve $P_1 P_2 P_3 Q_3 Q_2 Q_1 P_1$. The set $P_1 A P_3 + Q_1 B Q_3$ is exterior to $I_{J_1} + J_1$, since $I_{J_1} + J_1$ is a subset of $I_J + J$ and A and B are limit points of E_J. The set I_{J_1} contains a point L. There exists D_L of I_{J_1} and every point of J_1 is a limit point of D_L. Then, there is an arc $L_1 L_2 L_3$ such that L_1 and L_3 lie on $\underline{P_1 P_2 P_3}$ and $\underline{Q_1 Q_2 Q_3}$ respectively, and $\underline{L_1 L_2 L_3}$ is a subset of I_{J_1} and, consequently, of I_J. Hence we have an arc in I_J joining a point of D_P to a point of D_Q contrary to the assumption made above. But P and Q were any points of I_J. Then I_J is connected, in fact, connected in the strong sense.

Theorem 6. *If R is a region and ABC an arc of a closed curve J such that A and C are on the boundary of R and $\underline{ABC}$ is a subset of R, and if M is the set of all points that can be joined to a point of $\underline{ABC}$ by an arc whose every point, except an end-point on ABC, is common to I_J and R, then M is a connected set of points.*

Proof. Let P and Q be any two points of M and PP_1 and $Q Q_1$ two arcs common to I_J and R except for end-points P_1 and Q_1 which belong to $\underline{ABC}$. If $\underline{PP_1}$ and $\underline{QQ_1}$ have a common point, P and Q being any two points of M, the theorem is proved. Suppose that $\underline{PP_1}$ and $\underline{Q Q_1}$ have no common points and that P_1 is distinct from Q_1. If $P_1 \equiv Q_1$, the proof follows, with suitable modifications, from that given below. Since I_J is connected, there is an arc PXQ such that PXQ is a subset of I_J. A subset of $PP_1 +$

[1]) This arc lies in I_{J_1} except for end-points.

$+PXQ+QQ_1$ is an arc P_1YQ_1 such that $\underline{P_1YQ_1}$ belongs to I_J. Suppose that $\underline{P_1YQ_1}$ contains at least one point of $S-R'$, and let T_1 and T_2 be the first and last points respectively that P_1YQ_1 has in common with the boundary of R. (If P_1YQ_1 contains no point of $S-R'$, but does contain points of the boundary of R, the theorem follows immediately, provided such an arc can be obtained in connection with any two points P and Q of M). Let B_1 be a point of ABC between P_1 and Q_1. Let J_1 denote the closed

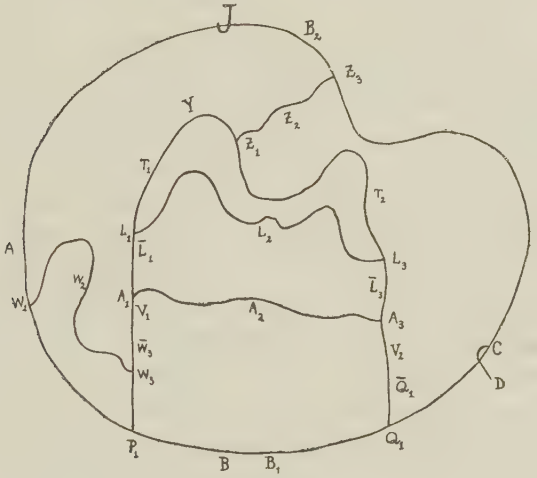

Fig. 1.

curve $P_1B_1Q_1YP_1$. The set I_{J_1} is a subset of I_J. There exists an arc $L_1L_2L_3$ such that L_1 and L_3 lie on J_1 in the order $P_1L_1T_1YT_2L_3Q_1$ and $\underline{L_1L_2L_3}$ is a subset of I_{J_1}. Suppose that $L_1L_2L_3$ contains a point of $S-R'$ (see remark above). Let J_2 denote the closed curve $P_1B_1Q_1L_3L_2L_1P_1$. Then, I_{J_2} is a subset of I_{J_1} and, consequently, of I_J. Let V_1 and V_2 be two points on J_2 in the order $P_1V_1L_1L_2L_3V_2Q_1$. Let J_3 denote the simple closed curve $P_1T_1YT_2Q_1B_2AP_1$, where B_2 is a point on J which with B_1 separates A and C on J. Then, $I_J=I_{J_1}+I_{J_3}+\underline{P_1T_1YT_2Q_1}$ (Moore, Theorem 25. The proof of this theorem follows without modification if the interior of a simple closed curve is now interpreted to be an I_J-set of points). There exists an arc $W_1W_2W_3$ such that $\underline{W_1W_2W_3}$ is a subset of I_{J_3} and W_1 and W_3 lie on J_3 in the order

$AW_1P_1W_3V_1$. Let J_4 denote the closed curve $W_1W_2W_3 + W_3P_1$ (on $P_1V_1T_1$) $+ P_1W_1$ (on ABC). There exists a region R_C about C such that R'_C contains no point of $J_1 + J_4$. There is in R_C an arc CD joining C to a point D of E_J. There is a number δ such that D lies in $S - H'_\delta$, where H_δ is one of the regions of the sequence whose common part is $I_J + J$. There exists an arc $Z_1Z_2Z_3$ such that Z_1 lies on T_1YT_2, Z_3 lies on Q_1CB_2 between B_2 and C and $Z_1Z_2Z_3$ lies in I_{J_3} and contains no point of $W_1W_2W_3$.

Consider J_2. Every boundary point of R lying in I_{J_2} is a limit point of the set of points of I_{J_2} in $S - R'$. Every point of I_{J_2} in $S - R'$ can be joined to some point of $L_1L_2L_3$ by an arc lying in I_{J_2} and $S - R'$ (except for an end-point on $L_1L_2L_3$). Then $L_1L_2L_3$ together with the boundary points of R lying in I_{J_2} and the points of I_{J_2} belonging to $S - R'$ form a closed connected set. Now form α_1 and α_2 of Axiom 8, using only regions that lie in I_J and contain no point or boundary point in common with $J_4 + CD + P_1W_3V_1 + P_1B_1Q_1V_2$. The regions R_1 and R_2 are to be suitably chosen regions containing L_1 and L_3 respectively. The set $\alpha_1 - R_1 - R_2$ is a finite set of regions covering points of $L_1L_2L_3$ not in $R_1 + R_2$ and all points of the boundary of R that belong to I_{J_2} and all points of I_{J_2} lying in $S - R'$ such that no region of the set has a point or boundary point on $L_1T_1YT_2L_3$. The set $\alpha_2 - R_1 - R_2$ is a finite set of regions covering points of $L_1T_1YT_2L_3$ not in $R_1 + R_2$ such that no region of the set has a point or boundary point in common with a region of $\alpha_1 - R_1 - R_2$. Such sets of regions evidently exist so that α_1 and α_2 satisfy Axiom 8. By this axiom, there is a region H containing $\alpha_1 + \alpha_2$ with a boundary which is a subset of the boundary of $\alpha_1 + \alpha_2$. The simple closed curve J_4 evidently lies in $S - H'$, since for each point of J_4 there is an arc joining this point to D and containing no point on the boundary of H. In a similar manner there may be obtained a region H_1 containing $J_4 + P_1B_1Q_1$ such that H_1 lies in H_δ, $Z_1Z_2Z_3 + V_1L_1T_1 + V_2L_3T_2 + CD + BZ_3C + H'$ lies in $S - H'_1$, and H'_1 lies in $S - H'$. By the reasoning employed in Theorem 5, it can be shown that there exists an arc $A_1A_2A_3$ such that A_1 and A_3 lie on $W_3V_1L_1$ and $Q_1V_2L_3$ respectively and $A_1A_2A_3$ is a subset of the common part of I_{J_2} and R. (In this case, however, the points of the classes C_1, C_2, C_3, C_4 are subsets of the set of points of I_{J_2} not $H + H_1$. The arcs $\overline{L_1V_1W_3}$ and $\overline{L_3V_2Q_1}$ respectively corres-

pond to the arcs T_1PT_2 and V_1YV_2 mentioned in Theorem 5, where $\overline{L_1}$ and $\overline{W_3}$ are points of $\underline{L_1V_1W_3}$, $\overline{L_3}$ and $\overline{Q_1}$ are points of $\underline{Q_1V_2L_3}$, and the arcs $\overline{L_1}L_1L_2L_3\overline{L_3}$ and $\overline{W_3}W_3P_1B_1Q_1\overline{Q_1}$ are subsets of H and H_1 respectively. The arcs used in defining C_i $(i = 1, 2, 3, 4)$ are to lie in the common part of I_{J_2} and R with the exception of a single end-point in each instance). But A_1 can be joined to P and A_2 to Q by arcs lying in M. Then, P can be joined to Q by an arc lying in M. Therefore, M is connected.

Theorem 7. *The boundary of every region is a simple closed curve.*

Proof. Let C and D be any two points of the boundary of a region R. By Axioms 2, 3, 6, there exists a simple closed curve CY_1DY_2C such that the arcs CY_1D and CY_2D belong to R and $S - R'$ except for the points C and D. Let $\overline{J}$ denote the simple closed curve CY_1DY_2C. Then, $I_{\overline{J}}$ contains a non-vacuous subset M of the boundary of R. Suppose that M is not connected. Then, $M = M_1 + M_2$, where M_1 and M_2 are two sets such that neither set contains a limit point of the other set. Let P_1 be a point of M_1. There exists an arc $Z_1P_1W_1$ such that Z_1 and W_1 belong to $\underline{CY_1D}$ and $\underline{CY_2D}$ respectively, $\underline{Z_1P_1W_1}$ is a subset of $I_{\overline{J}}$ and P_1 is the only point of M on $Z_1P_1W_1$. Let $\overline{J_1}$ and $\overline{J_2}$ denote the simple closed curves $Z_1P_1W_1CZ_1$ and $Z_1P_1W_1DZ_1$ respectively. Then, $I_{\overline{J}} = I_{\overline{J_1}} + I_{\overline{J_2}} + \underline{Z_1P_1W_1}$ (Moore, Theorem 25). There exists a point P_2 of M_2 in either $I_{\overline{J_1}}$ or $I_{\overline{J_2}}$. Suppose that P_2 belongs to $I_{\overline{J_2}}$. There is a region $\overline{R}$ about P_2 such that $\overline{R}'$ lies in $I_{\overline{J_2}}$. By Axiom 6 there is an arc $A_2P_2B_2$ in $\overline{R}$ such that A_2 and B_2 belong to R and $S - R'$ respectively and P_2 is the only point that this arc has on the boundary of R. There exist arcs Y_1A_2 in R and Y_2B_2 in $S - R'$ such that Z_2 and W_2 respectively are the first points that these arcs have on $\overline{J_2}$. A subset of $Y_1A_2 + A_2P_2B_2 + Y_2B_2$ is an arc $Z_2P_2W_2$ such that $\underline{Z_2P_2W_2}$ is a subset of $I_{\overline{J_2}}$. Suppose that Z_2 and W_2 lie on $\overline{J}$ in the order $Z_1Z_2DW_2W_1$. (If one or both of the points Z_2 and W_2 lie on $Z_1P_1W_1$, the following argument still holds). Let A and B be two points of $\overline{J}$ in the order $Z_1AZ_2DW_2BW_1$. Let J denote the simple closed curve $Z_1AZ_2P_2W_2BW_1P_1Z_1$. The set I_J contains a subset of M. Suppose that $\overline{M_1}$ and $\overline{M_2}$ respectively are non-vacuous subsets of M_1 and M_2 in I_J (I_J is a subset of $I_{\overline{J}}$). If one

of these sets is vacuous, it will be obvious that the desired result still obtains. In the following proof the sets $\underline{Z_1 C W_1}$ and $\underline{Z_2 D W_2}$ belonging to $\overline{J}$ are to be disregarded in naming arcs. Thus, AP_2B means the arc AP_2B of J. There exists a region R_B containing B and no point of R'. In R_B there is an arc BE joining B to a point E of E_J and having B_1 as the last point on J. There is a number δ such that E lies in $S - H'_\delta$ where H_δ is a region of the sequence of regions whose common part is $I_J + J$. There is a region R_{P_1}

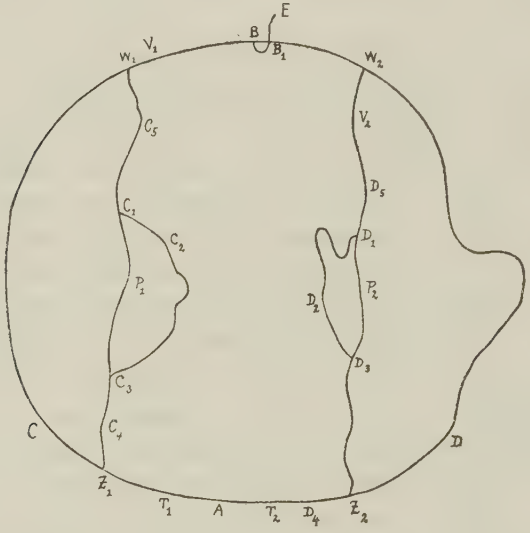

Fig. 2.

about P_1 such that R'_{P_1} lies in H_δ and contains no point of $B_1E + \overline{M_2} + AP_2B$. As an almost immediate consequence of Theorem 6, it can be shown that there is an arc $C_1 C_2 C_3$ such that $\underline{C_1 C_2 C_3}$ is common to R_{P_1} and I_J, and C_1 and C_3 lie on J in the order $C_1 P_1 C_3 A$. Let J_1 denote the simple closed curve $C_1 C_2 C_3 P_1 C_1$. No point of $\overline{M_1}$ lies in I_{J_1} since I_{J_1} is a subset of R_{P_1}. Cover each point and boundary point of the subset of M_1 not in I_{J_1} (P_1 is not a limit point of this set) by a region which, with its boundary, lies in I_J and has no point or boundary point in common with $\overline{M_2}$. A finite subset of these regions, $[K]$, will cover the given closed set. Consider any region K_1 of $[K]$. Such a region has a point Q_1 in R.

There is in R an arc joining Q_1 to C_3 having $\overline{Q_1}$ as a first point on the set composed of the arcs $C_1C_2C_3$ and P_1AP_2. There exist arcs of this type for each region of $[K]$. Since $[K]$ is a finite set, all of the end-points $\overline{Q_i}$ lie on $C_1C_2C_3$ or P_1C_3D, where D is a point on P_1AP_2 between C_3 and P_2. Form α_1 and α_2 of Axiom 8 as follows: R_1 and R_2 are regions about C_1 and P_1 respectively. The set of regions $\alpha_1 - R_1 - R_2$ is a finite set covering all points of arc P_1C_1 (of $P_1C_1B_1$) not in $R_1 + R_2$. No region of $\alpha_1 - R_1 - R_2$ is to have a point or boundary point on $P_1AP_2B_1C_1 + B_1E + C_1C_2C_3$. The set $\alpha_2 - R_1 - R_2$ consists of $[K]$ and a finite set of regions covering all points of $C_1C_2C_3 + P_1C_3D +$ all arcs $Q_i\overline{Q_i}$ not in $[K] + R_1 + R_2$. All regions used in obtaining $\alpha_1 + \alpha_2$ are to lie in H_δ and to have no point or boundary point in common with $B_1E + \overline{M_2} + P_2B_1$ (of $P_2B_1P_1$). Obviously, there exist regions as indicated which form α_1 and α_2 in satisfaction of the requirements of Axiom 8. According to the same axiom, there is a region containing $\alpha_1 + \alpha_2$ whose boundary is a subset of the boundary of $\alpha_1 + \alpha_2$. Denote this region by R_{P_1}. There exists a region R_{P_2} lying in H_δ and containing P_2 such that R'_{P_2} contains no point of $P_1C_1B_1E + \overline{R'_{P_1}}$. As before, there is an arc $D_1D_2D_3$ such that $D_1D_2D_3$ is common to I_J and R_{P_2}, and D_1 and D_3 lie on J in the order $D_3P_2D_1B_1$. Cover each point of $\overline{M_2}$ with a region which, together with its boundary, lies in I_J and has no point or boundary point in common with any region of $\alpha_1 + \alpha_2$. Cover P_2 with a region belonging to H_δ and containing no point or boundary point in common with $\overline{R'_{P_1}} + P_1C_1B_1E + D_1D_2D_3$. A finite subset, G, of all these regions, including the region about P_2, will cover the closed set $\overline{M_2} + P_2$. If any region of G contains a point of $\overline{R_{P_1}}$, then this region lies wholly in $\overline{R_{P_1}}$. Let G_1 be the set of all regions of G that lie in the exterior of $\overline{R_{P_1}}$. The set G_1 is not vacuous, as the region used to cover P_2 belongs to G_1. Let $\overline{G_1}$ be the set of all regions of G_1 which, together with their boundaries, form with $D_1D_2D_3$ a connected set. If there is any region of $\overline{G_1}$ that has only a set of boundary points in common with other regions of $\overline{G_1}$ and no point or boundary point on $D_1D_2D_3$, add to $\overline{G_1}$ a finite set of regions covering these boundary points such that each region of this additional set lies in I_J and has no point or boundary point

in common with $\overline{R'_{P_1}} + J$ or with any region of G_1 not in $\overline{G_1}$. These additional regions evidently exist in finite number covering the set of boundary points mentioned, since this set of boundary points is closed. Denote by $[\overline{K}]$ the augmented set of regions which includes $\overline{G_1}$. Form $\bar{\alpha}_1$ and $\bar{\alpha}_2$ as follows, using only regions that lie in H_δ and have no point or boundary point in common with $\overline{R'_{P_1}} + P_1 C_1 B_1 E$ or with any region of G_1 not in $\overline{G_1}$. Let $\overline{R_1}$ be the region of G_1 containing P_2, $\overline{R_2}$, a region containing D_3, $\bar{\alpha}_1 - \overline{R_1} - \overline{R_2}$, a finite set of regions containing $[\overline{K}]$ and all points of $P_2 D_1 D_2 D_3$ not in $[\overline{K}] + \overline{P_1} + \overline{R_2}$, $\bar{\alpha}_2 - \overline{R_1} - \overline{R_2}$, a finite set of regions covering all points of $P_2 D_3$ (of $P_2 D_3 A$) not in $\overline{R_1} + \overline{R_2}$. Obviously, $\bar{\alpha}_1$ and $\bar{\alpha}_2$, consisting of regions of this character and satisfying the requirements of Axiom 8, can be obtained. By Axiom 8, there is a region containing $\bar{\alpha}_1 + \bar{\alpha}_2$ whose boundary is a subset of the boundary of $\bar{\alpha}_1 + \alpha_2$. Let $\overline{R_{P_2}}$ denote this region. Every point on the boundary of $\overline{R_{P_1}}$ can be joined to E by an arc containing no point of the boundary of $\overline{R_{P_2}}$. Then, the boundary of $\overline{R_{P_1}}$ lies in $S - \overline{R'_{P_2}}$. If $\overline{R_{P_1}}$ had a point P_3 in common with $\overline{R_{P_2}}$, an arc in $\overline{R_{P_2}}$ joining P_3 to P_2 would contain a point on the boundary of $\overline{R_{P_1}}$. Thus, it can be shown that $\overline{R'_{P_1}}$ and $\overline{R'_{P_2}}$ are mutually exclusive. Let G_2 be the subset of G in the common exterior of $\overline{R_{P_1}}$ and $\overline{R_{P_2}}$. Let C_4 and C_5 respectively be the last points that the arcs $P_1 A P_2$ and $P_1 B P_2$ have in common with the boundary of $\overline{R_{P_1}}$. Let $R_1^{(1)} \equiv R_1$ (see above), $R_2^{(1)} \equiv R_2$, $\alpha_1^{(1)} - R_1^{(1)} - R_2^{(1)} \equiv \alpha_1 - R_1 - R_2$. The boundary points of $\overline{R_{P_1}}$ that belong to $I_J + J$ form a closed bounded set. Hence, there exists a finite set of regions covering this set of boundary points such that no region of this set has a point or boundary point in common with $\overline{R'_{P_2}} + (G_2)' + J_1 + P_2 D_1 B_1 E + (\alpha_1 - R_1 - R_2)'$. Let R_β be a region of this finite set that has no point or boundary point on $C_5 C_1 C_2 C_3 C_4$. The region R_β contains a point Q_β of $\overline{R_{P_1}}$. There is in $\overline{R_{P_1}}$ an arc joining Q_β to C. This arc has a first point $\overline{Q_\beta}$ on $C_5 C_1 C_2 C_3 C_4$. Cover each point of every such arc $Q_\beta \overline{Q_\beta}$ not in R_β by a region which, with its boundary, lies in R_{P_1} and contains no point or boundary point in common with $\alpha_1 - R_1 - R_2$. A finite subset, including the regions R_β will cover the arcs $Q_\beta \overline{Q_\beta}$

Let $[\overline{R}]$ denote this finite set of regions. The set $\alpha_2^{(1)} - R_1^{(1)} - R_2^{(1)}$ is to consist of $\alpha_2 - R_1 - R_2 + [\overline{R}] +$ a finite set of regions covering all points of $C_5 C_1 C_2 C_3 C_4$ not in $\overline{R}_{P_1} + [\overline{R}]$. This last-mentioned finite set of regions will contain only regions that have no point or boundary point in common with $\overline{R}_{P_2} + G_2$ or with the regions of $\alpha_1^{(1)} - R_1^{(1)} - R_2^{(1)}$. The set $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$, as specified above, can evidently be obtained so as to meet the requirements of Axiom 8. Let $R_{P_1}^{(1)}$ denote the corresponding region. $R_{P_1}^{(1)}$ contains $\overline{R}_{P_1}$ and lies in $S - \overline{R}_{P_2}$. By a similar process, we can obtain a region $R_{P_2}^{(1)}$ which contains $\overline{R}_{P_2}$ and all boundary points of $\overline{R}_{P_2}$ in $I_J + J$ and all points of $D_4 D_3 D_2 D_1 D_5$ not in $\overline{R}_{P_2}$, where D_4 and D_5 are the last points that arcs $P_2 A P_1$ and $P_2 B P_1$ have on the boundary of $\overline{R}_{P_2}$. Furthermore, the additional regions used in obtaining $R_{P_2}^{(1)}$ have no point or boundary point in common with $R_{P_1}^{(1)} + G_2$, and every point or boundary point of one of the regions, $R_{P_1}^{(1)}$, $R_{P_2}^{(1)}$, can be joined to E by an arc lying in the exterior of the other region. G_3 is now the subset of G_2 in the common exterior of $R_{P_1}^{(1)}$ and $R_{P_2}^{(1)}$. Let T_1, V_1, T_2, V_2 be points of J in the order $V_1 C_5 P_1 C_4 T_1 T_2 D_4 P_2 V_2$ such that the arcs $T_1 P_1 V_1$ and $T_2 P_2 V_2$ belong to $R_{P_1}^{(1)}$ and $R_{P_2}^{(1)}$ respectively. Evidently the arcs $T_1 T_2$ (of $P_1 A P_2$) and $V_1 V_2$ (of $P_1 B P_2$) are in the common exterior of $\overline{R}_{P_1}$ and $\overline{R}_{P_2}$. By the reasoning employed in Theorem 5, which in this case would involve a classification of the points of I_J not in $R_{P_1}^{(1)} + R_{P_2}^{(1)}$, we can obtain an arc $L_1^{(1)} L_2^{(1)} L_3^{(1)}$ such that $L_1^{(1)} L_2^{(1)} L_3^{(1)}$ lies in I_J and contains no point of $\overline{R}_{P_1} + \overline{R}_{P_2}$, and $L_1^{(1)}$ and $L_3^{(1)}$ lie on $T_1 T_2$ (of $P_1 A P_2$) and $V_1 V_2$ (of $P_1 B P_2$) respectively. If $L_1^{(1)} L_2^{(1)} L_3^{(1)}$ contains no point or boundary point of a region of G_2 then we have an arc from a point $L_1^{(1)}$ of R to a point $L_3^{(1)}$ of $S - R'$ which contains no point of the boundary of R. This contradiction would exhibit the falsity of the assumption that the subset of the boundary of R which lies in I_J is not connected. Suppose, however, that $L_1^{(1)} L_2^{(1)} L_3^{(1)}$ contains points or boundary points belonging to regions of G_2. Denote by J_3 the simple closed curve $C_1 P_1 L_1^{(1)} L_2^{(1)} L_3^{(1)} C_1$. The point C_2 belongs to I_{J_3} (Moore, Theorem 25). Suppose that R_α is a region of G_2 containing a point or boundary point on $L_1^{(1)} L_2^{(1)} L_3^{(1)}$. In the first case, R_α contains a point Q_α of I_{J_3}. In the second case, there is a region $\overline{R}_\alpha$ containing each boundary point of the region R_α which belongs to

$L_1^{(1)} L_2^{(1)} L_3^{(1)}$ such that $\overline{R}_\alpha$ lies in I_J and contains no point of $\overline{R}_{P_2}$. In either case, there is a point Q_α of I_{J_3} belonging to R_α or $\overline{R}_\alpha$ as the case may be. In I_{J_3} there is an arc $Q_\alpha C_2$ with $\overline{Q}_\alpha$ as the first point on $C_1 C_2 C_3$. Since the set of boundary points of the regions of G_2 lying on $L_1^{(1)} L_2^{(1)} L_3^{(1)}$ is closed, there exists a finite set of regions of the type of $\overline{R}_\alpha$ covering all boundary points of regions of G_2 that lie on $L_1^{(1)} L_2^{(1)} L_3^{(1)}$. Let X be the set of all regions of G_2 that have a point or boundary point on $L_1^{(1)} L_2^{(1)} L_3^{(1)}$ and all regions (finite in number) of the type of $\overline{R}_\alpha$ described above. Let $\overline{X}$ be the set of regions of G_2 not contained in X that have a point or boundary point in common with a region of X. If a region of $\overline{X}$ has only boundary points in common with the region of X, the set of all such boundary points is closed. There exists a finite set of regions covering these boundary points such that each region is in I_J and contains no point or boundary point in common with $\overline{R}_{P_2}$ or any region of G_2 not in $X + \overline{X}$. Let X_1 be the set of regions consisting of these additional regions together with $X + \overline{X}$. Form $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ as follows: $R_1^{(2)} \equiv R_1^{(1)}$, $R_2^{(2)} \equiv R_2^{(1)}$, $\alpha_1^{(2)} - R_1^{(2)} - R_2^{(2)} \equiv \equiv \alpha_1^{(1)} - R_1^{(1)} - R_2^{(1)}$.

The set $\alpha_2^{(2)} - R_1^{(2)} - R_2^{(2)}$ consists of $\alpha_2^{(1)} - R_1^{(1)} - R_2^{(1)}$, X_1 and a finite set of regions lying in I_{J_3} together with their boundaries and covering all points not in $\alpha_2^{(1)} + X_1$ of arcs of the type of $Q_\alpha \overline{Q}_\alpha$. Evidently $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ exist in such wise as to satisfy Axiom 8. Denote by $R_{P_1}^{(2)}$ the corresponding region postulated by the axiom. The region $R_{P_1}^{(2)}$ has no point or boundary point in common with $\overline{R}_{P_2}$. Proceeding as before, with the help of two additional regions containing $R_{P_1}^{(2)}$ and $\overline{R}_{P_2}$ respectively, we obtain an arc $L_1^{(2)} L_2^{(2)} L_3^{(2)}$ such that $L_1^{(2)} L_2^{(2)} L_3^{(2)}$ lies in I_J and contains no point of $R_{P_1}^{(2)} + \overline{R}_{P_2}$, and $L_1^{(2)}$ and $L_3^{(2)}$ belong to $P_1 A P_2$ and $P_1 B P_2$ respectively. This process may be continued, but will end after a finite number of steps as the number of regions in G is finite. There is obtained finally an arc $L_1^{(n)} L_2^{(n)} L_3^{(n)}$ joining a point $L_1^{(n)}$ of $P_1 A P_2$ to a point $L_3^{(n)}$ of $P_1 B P_2$ and containing no point of the boundary of R. If one of the sets, $\overline{M}_1$ or $\overline{M}_2$, is vacuous, obviously the preceding argument may be modified so as to lead to a contradiction in this case. Hence M, the subset of the boundary of R in $I_{\overline{J}}$ is connected. The interior of any simple closed curve related to R as

J is related to R contains a connected subset of the boundary of R. The same type of reasoning will show that both P_1 and P_2 are limit points of the connected set $\overline{M}$, where $\overline{M}$ is the subset of the boundary of R in I_J. The set $P_1 + \overline{M} + P_2$ is a closed bounded connected set of points. It can be easily proved that the omission of any point of $\overline{M}$ disconnects the set. Hence $P_1 + \overline{M} + P_2$ is an arc from P_1 to P_2.

Let $P_1 P_3 P_2$ be the arc which is the subset of the boundary of R in $I_J + J$. Denote by N the subset of the boundary of R which lies in E_J. If Q be any point of N, by Axioms 2, 3 and 6, there is an arc $F\,Q\,K$ such that F and K belong to $P_1 A P_2$ and $P_1 P_3 P_2$ respectively and $\overline{QF}$ and $\overline{QK}$ lie in R and $S - R'$ respectively. By Theorem 27, Moore, which now holds, either P_2 lies in the exterior of the simple closed curve $FQKP_1F$ or P_1 lies in the exterior of the simple closed curve $FQKP_2F$. If the first alternative is true, it can be shown, by an argument which makes use of the type of reasoning used before, that the subset of N within $FQKP_1F$ consists of points of an arc joining P_1 to Q. Then for any point Q of N, there is an arc joining Q to P_1 or P_2 and consisting of points of N except for the points P_1 or P_2.

(1) Suppose that there exists no arc joining a point Q of N to P_2 and consisting only of points of the boundary of R and let $\overline{Q}$ be any fixed point of N. Let $\overline{F}\,\overline{Q}\,\overline{K}P_1\overline{F}$, denoted by $\overline{J}$, be a simple closed curve of the nature of $FQKP_1F$ described above. Then, N consists of $P_1\overline{Q} - P_1$ together with the subset of N in $E_{\overline{J}}$. Let Q_δ be a point of N in $E_{\overline{J}}$. There exists the corresponding simple closed curve $F_\delta\,Q_\delta\,K_\delta P_1 F_\delta$, denoted by J_δ, where F_δ lies on $P_1 A P_2$ and belongs to $E_{\overline{J}}$. The set I_{J_δ} contains a subset of N consisting of points of an arc joining P_1 to Q_δ. Evidently $\overline{Q}$ belongs to this arc. Thus every point of N in E_J can be joined to $\overline{Q}$ by an arc consisting of points of N. Then $P_1 + N$ is connected. If P_2 is a limit point of $P_1 + N$, then $P_1 + N + P_2$ is a closed bounded connected set of points. It can easily be shown that this set is disconnected by the omission of any point other than P_1 or P_2. Under the circumstances, $P_1 + N + P_2$ is an arc from P_1 to P_2 and the boundary of R is proved to be a simple closed curve. Suppose, however, in addition to our first assumption, we assume that P_2 is not a limit

point of $P_1 + N$. Then, $P_1 + N$ is closed. Keep $\overline{J}$ fixed. Then, there exists a simple closed curve $J_{\overline{Q}}$ (J_δ of the preceding discussion) such that $\overline{Q}$ is a subset of $I_{J_{\overline{Q}}}$. If Q_μ is any point of N in $E_{\overline{J}}$ a simple closed curve J_{Q_μ} of like character can be found such that Q_μ is a subset of $I_{J_{Q_\mu}}$. Assign to each Q_μ a definite simple closed curve of this character. Cover $\overline{Q}$ with a region lying in $I_{J_{\overline{Q}}}$. Cover each point Q_μ in $E_{\overline{J}}$ with a region lying in the corresponding $I_{J_{Q_\mu}}$. The set consisting of $\overline{Q}$ and the subset of N in $E_{\overline{J}}$ is closed. Then a finite subset of the regions just mentioned will cover the same closed set. But, corresponding to each region of the finite set of

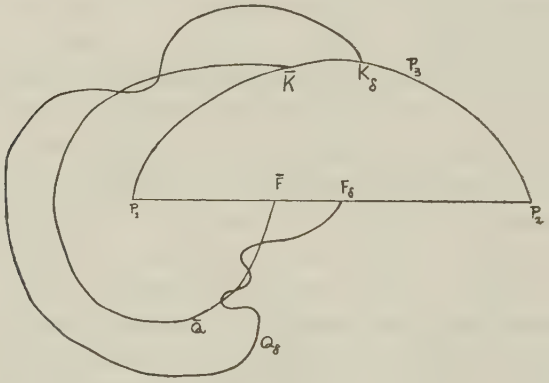

Fig. 3.

regions there is an open set $I_{J_{Q_\mu}}$ containing the region. Then $\overline{Q} + \{Q_\mu\}$ is covered by a finite number of the open sets $I_{J_{Q_\mu}}$. Let $[O]$ denote this finite set of open sets. Then,

$$[O] \equiv \{I_{J_{Q_\mu^{(1)}}}, I_{J_{Q_\mu^{(2)}}}, \ldots, I_{J_{Q_\mu^{(n)}}}\}.$$

By the preceding discussion, $\overline{Q}$ is contained in each $I_{J_{Q_\mu^{(i)}}}$. Let $\overline{Q}_\mu^{(i)}$ be the single point of N on $J_{Q_\mu^{(i)}}$. If there is no point $Q_\mu^{(i)}$ which lies in $E_{J_{Q_\mu^{(1)}}}$, then, there is an arc $P_1\,\overline{Q}_\mu^{(1)}$ which contains all points of N, and $\overline{Q}_\mu^{(1)}$ does not belong to any set of $[O]$, which fact furnishes a contradiction. Suppose, however, that $\overline{Q}_\mu^{(\beta)}$ is the first point of $\{\overline{Q}_\mu^{(i)}\}$ which lies in $E_{J_{Q_\mu^{(1)}}}$. Then, all points of N in $I_{J_{Q_\mu^{(\beta)}}}$ are

points of an arc joining P_1 to the point $\overline{Q}_\mu^{(\beta)}$. If there are no points $\overline{Q}_\mu^{(i)}$ in $E_{J_{\overline{Q}_\mu^{(\beta)}}}$, we reach a contradiction as before. As the points of $\{\overline{Q}_\mu^{(i)}\}$ are finite in number, we must reach a final contradiction. Thus, if no point of N can be joined to P_2 by an arc consisting of points of N except for P_2, the point P_2 must be a limit point of N and the boundary of R is a simple closed curve. If P_1 takes the place of P_2 in the preceding argument, we arrive at the same result.

(2) Assume that there are points of N that can be joined to P_1 and points of N that can be joined to P_2 by arcs which lie in N except for P_1 or P_2, as the case may be. Let N_{P_1} and N_{P_2} respectively denote these subsets of N. If N_{P_1} and N_{P_2} have a common point, or if one set contains a limit point of the other, then, $P_1 + N + P_2$ is a closed bounded connected set which is disconnected by the omission of one point other than P_1 or P_2. In this case, the boundary of R is shown to be a simple closed curve. If $P_1 + N_{P_1}$ and $P_2 + N_{P_2}$ are mutually exclusive closed sets, then, by considering $P_1 + N_{P_1}$, we are led to a contradiction as in (1). Therefore, the boundary of any region R is a simple closed curve.

Theorem 8. (Moore, Theorem 28). *If P is a point of a simple closed curve J and R_P, a region about P, there exists a simple continuous arc AXB such that* (1) *A and B are on J,* (2) *$\underline{AXB}$ is common to R_P and I_J (E_J),* (3) *of the two arcs into which A and B divide J, that one which contains P lies in R_P.*

Proof. That there exists an arc $A_1X_1B_1$ of the character described such that $\underline{A_1X_1B_1}$ is common to I_J and R_P follows immediately from Theorem 6. Let J_{R_P} be the simple closed curve which is the boundary of R_P (Theorem 7). There exists an arc CA_1PB_1D of J such that $\underline{CA_1PB_1D}$ is a subset of R_P and C and D lie on J. The points C and D divide J_{R_P} into two arcs CT_1D and CT_2D. Let J_1 and J_2 respectively denote the simple closed curves CT_1DPC and CT_2DPC. If J_3 is the simple closed curve $A_1X_1B_1PA_1$, then I_{J_3} is a subset of I_{J_1} or I_{J_2} (Moore, Theorem 25). Suppose that I_{J_3} is a subset of I_{J_1}. Let $\overline{R}_P$ be a region containing P such that $\overline{R}_P'$ is a subset of R_P and contains no point of $A_1X_1B_1 + J - \underline{A_1PB_1}$. By the first part of the proof, there is an arc of the character described in the theorem relative to J_2 and $\overline{R}_P$. If $A_2X_2B_2$ denotes this arc, then $A_2X_2B_2$ is common to I_{J_2} and $\overline{R}_P$. The set $\underline{A_2X_2B_2}$ cannot

belong to I_J. For, if $\underline{A_2X_2B_2}$ belonged to I_J, it would be a subset of I_{J_3}, and I_{J_3} belongs to I_{J_1}. Hence $A_2X_2B_2$ is an arc of the character described in the theorem and $\underline{A_2X_2B_2}$ belongs to E_J and R_P.

Theorem 9. *A simple closed curve is accessible from all sides at every point.*

This theorem can be proved without difficulty by means of Theorem 8 and the reasoning employed by Moore, page 148.

Remark. All of the fifty-two theorems of the paper by Moore, not already proved in the preceding, can now be demonstrated precisely as proved by Moore except that the interior of a simple closed curve is to be interpreted as an I_J-set of points. The results of the preceding theorems can now be expressed in the following fundamental theorem.

Theorem 10. *If J is a simple closed curve, there exist two sets, I_J and E_J, such that* (1) *J is the common boundary of I_J and E_J,* (2) *any two points of I_J (E_J) can be joined by a simple continuous arc lying wholly on I_J (E_J),* (3) *I_J is a bounded, and E_J, an unbounded, set and* (4) *J is accessible from all sides at every point.*

Theorem 11. *Every simple closed curve is the boundary of a set of points that has all the properties of a region.*

Proof. The proof of this theorem consists in showing that all of the axioms hold if a region is interpreted to be an I_J-set of points. Theorems 1, 7, 10 show that Axioms 1, 2, 3, 4, 6, 7 are satisfied. Axiom 5, of course, holds.

In Axiom 8, let I_J-sets of points replace regions. Denote by I_{J_1}, I_{J_2}, $\bar{\alpha}_1$, $\bar{\alpha}_2$ the I_J-sets of points corresponding to R_1, R_2, α_1, α_2 respectively. By Moore, Theorem 41, there exists I_{J_μ}, an I_J-set of points corresponding to the region H of Axiom 8. There exists a simple closed curve $\overline{J}$ such that $\overline{J}$ consists of two arcs $P_1T_1P_2$ and $P_1T_2P_2$ where P_1, P_2, $P_1T_1P_2$, $P_1T_2P_2$ belong to I_{J_1}, I_{J_2}, $\bar{\alpha}_1$, $\bar{\alpha}_2$ respectively. The set $I_{\overline{J}}$ is a subset of I_μ. There exists a region H_1 containing J_1 such that H_1' has no point in common with $I_{J_2}+J_2$. The boundary J_{H_1} of H_1 is a simple closed curve. J_{H_1} contains no point common to $\bar{\alpha}_1$ and $\bar{\alpha}_2$. It can be shown that there exists an arc AB belonging to J_{H_1} such that $\underline{AB}$ is a subset of $I_{\overline{J}}$ and A and B lie on $\overline{J}$ and belong to $\bar{\alpha}_1 - I_{J_1} - I_{J_2}$ and $\bar{\alpha}_2 - I_{J_1} - I_{J_2}$ respectively. The arc AB may be identified with the set M of Axiom 8.

Theorem 12. *Every set of points satisfying Axioms I constitutes a space homeomorphic with the Euclidean plane.*

In a paper [1]), „*Concerning a set of postulates for plane analysis situs*“, R. L. Moore has proved that every set of points satisfying his Σ_1 system of axioms is homeomorphic with the Euclidean plane. It can be shown that all the axioms of this set not included in our system are satisfied by a set of points satisfying Axioms I. Furthermore, Axiom 8 holds in the plane.

Axioms II. This set of axioms differs from the preceding set in that Axioms 1 and 2 are replaced by the following axioms, which are Axioms 1′ and 2′ of Moore's set Σ_3.

Axiom 1′. *If P is a point, there exists an infinite sequence of regions, $R_1, R_2, R_3, \ldots$, such that* (1) *P is the only point that they have in common,* (2) *for every n, R_{n+1} is a proper subset of R_n,* (3) *if R is a region about P, then there exists n such that R_n' is a subset of R.*

Axiom 2′. *Every two points of a region are the extremities of at least one simple continuous arc that lies wholly in that region.*

Theorem 10′ (identical with Theorem 10).

Proof. (This proof is based upon Axioms II). Evidently there exist two finite sets of regions, $\bar{\alpha}_1$ and $\bar{\alpha}_2$, covering all points of J and satisfying the requirements of Axiom 8. Let $\overline{G}$ denote $\bar{\alpha}_1 + \bar{\alpha}_2$. In accordance with this axiom, there exists a region $\overline{H}$ such that $\overline{H}$ contains $\bar{\alpha}_1 + \bar{\alpha}_2$ and the boundary of $\overline{H}$ is a subset of the boundary of $\bar{\alpha}_1 + \bar{\alpha}_2$. *All other finite sets of regions similar to $\bar{\alpha}_1$ and $\bar{\alpha}_2$ used hereafter to cover J are to lie with their boundaries in $\overline{G}$.*

(1) By Axiom 1′, there exists a definite sequence of regions for each point of J possessing the properties indicated in the axiom. Cover each point P of J with a region selected from the P-sequence of regions postulated by Axiom 1′ after the method of Theorem 1, and obtain the finite sets of regions $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ in accordance with Axiom 8 such that $\alpha_i^{(1)}$ is a subset of $\bar{\alpha}_i$. Let G_1 denote $\alpha_1^{(1)} + \alpha_2^{(1)}$. By Axiom 8, there exists a region H_1 corresponding to G_1. The region H_1 and its boundary is a subset of $\overline{H}$. Next cover each point P of J with a region of the P-sequence of Axiom 1′ of subscript greater than 2, such that each region with

[1]) Trans. Amer. Math. Soc. 20, 1919.

its boundary lies in G_1 and the finite subset of these regions which covers J is composed of two sets $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ in accordance with Axiom 8. The set $\alpha_i^{(2)}$ is to be a subset of $\alpha_i^{(1)}$ as in Theorem 1. Let G_2 denote $\alpha_1^{(2)} + \alpha_2^{(2)}$. There is a corresponding region H_2 as postulated by Axiom 8. Proceeding in this manner, there is obtained an infinite sequence of regions $H_1, H_2, H_3, \ldots$, each containing J, such that H_i' is a subset of H_{i-1}, and H_i' is a subset of the fundamental region $\overline{H}$ for every value of i. Let H_ω denote the common part of $H_1, H_2, H_3, \ldots$. The set H_ω is closed. Precisely as in Theorem 2, it can be shown that H_ω contains a point F not in $\overline{G}$.

(2) Let $[H_\omega]$ denote the totality of all such sets H_ω which may be obtained in the manner just indicated. Each H_ω contains J. Hence there is a set of points $\widehat{H}_\omega$ common to all the sets of $[H_\omega]$. A point P belongs to $\widehat{H}_\omega$ if and only if there is no set $H_\omega^{(\alpha)}$ of $[H_\omega]$ to which P does not belong. Since $\widehat{H}_\omega$ is a subset of the fundamental region $\overline{H}$, the set $\widehat{H}_\omega$ is bounded. Suppose that Q, a point of $\widehat{H}_\omega$ not on J, is a boundary point of $\widehat{H}_\omega$. There exists a region R_Q about Q such that R_Q' contains no point of J, and R_Q has within it a point Q_1 of $\widehat{H}_\omega$ and a point Q_2 of $S - \widehat{H}_\omega$. Since Q_2 lies in $S - \widehat{H}_\omega$, there exists a set $H_\omega^{(\beta)}$ of $[H_\omega]$ to which Q_2 does not belong. There is a region $H_\delta^{(\beta)}$ of the sequence of regions of which $H_\omega^{(\beta)}$ is the common part such that Q_2 belongs to the exterior of $H_\delta^{(\beta)}$. In the exterior of $H_\delta^{(\beta)}$ there is an arc $Q_2 E$ joining Q_2 to a point E of $S - \overline{H}'$. ($H_\delta^{(\beta)}$ lies in $\overline{H}$). The set $Q_2 E + R_Q'$ contains no point of J. There exists a set $H_\omega^{(\mu)}$ of $[H_\omega]$ such that $G_1^{(\mu)}$, which consists of the first α_1 and α_2 used in obtaining $H_\omega^{(\mu)}$, and the boundary of $G_1^{(\mu)}$ contains no point of $Q_2 E + R_Q'$. The corresponding region $H_1^{(\mu)}$ according to Axiom 8, has a boundary which is a subset of the boundary of $G_1^{(\mu)}$. Then, each point of $Q_2 E + R_Q'$ can be joined to E by an arc which contains no point of the boundary of $H_1^{(\mu)}$. The point E lies in the exterior of $\overline{H}$ and, therefore, in the exterior of $H_1^{(\mu)}$. Hence Q_1 does not belong to $H_\omega^{(\mu)}$ and, consequently, not to $\widehat{H}_\omega$. This contradiction shows that no point of $\widehat{H}_\omega$ not on J can belong to the boundary of $\widehat{H}_\omega$. Then, the boundary of $\widehat{H}_\omega$ is a subset of J. As in Theorem 4, it can be shown that every point of J is a boundary point of $\widehat{H}_\omega$. Since the boundary of $\widehat{H}_\omega$ belongs to $\widehat{H}_\omega$, the set $\widehat{H}_\omega$ is closed.

The set $S - H_\omega$ is connected. For, let P_1 and P_2 be two points of $S - H_\omega$. There exists at least one set $H_\omega^{(\alpha)}$ of $[H_\omega]$ to which P_1 does not belong Hence P_1 is exterior to a region $H_\delta^{(\alpha)}$, where $H_\delta^{(\alpha)}$ is a region of the sequence of regions whose common part is $H_\omega^{(\alpha)}$. There is in the exterior of $H_\delta^{(\alpha)}$ an arc P_1E joining P_1 to a point E of $S - \overline{H}'$. The arc P_1E contains no point of J, which, of course, belongs to $H_\delta^{(\alpha)}$. Similarly, there exists an arc P_2E containing no point of J. A subset of $P_1E + P_2E$ is an arc P_1P_2 containing no point of J, the boundary of $\overline{H}_\omega$. Then, any two points of $S - \overline{H}_\omega$ can be joined by an arc lying wholly in $S - \overline{H}_\omega$.

(3) As stated above, every set H_ω of $[H_\omega]$ contains at least one point F not belonging to $\overline{G}$. Let $[F]$ be the set of all such points. Since every point of $[F]$ belongs to $\overline{H}$, the set $[F]$ is bounded. No point of J, which belongs to $\overline{G}$, can be a limit point of $[F]$. Suppose that no point of $[F]$ belongs to $\overline{H}_\omega$. Cover each point Q of $[F]$ and its boundary with a region R_Q such that R'_Q contains no point of J. The simple closed curve J is the boundary of $\overline{H}_\omega$. Then, R'_Q lies in $S - \overline{H}_\omega$. A finite subset of the regions covering $[F]$ and its boundary will cover the same closed set. Since $S - \overline{H}_\omega$ is connected, there exists in $S - \overline{H}_\omega$ a finite set of arcs joining points of this finite set of regions (one point selected for each region) to a point E of $S - \overline{H}'$. These arcs, together with the finite set of regions and their boundaries, constitute a closed set N lying in $S - \overline{H}_\omega$. There exists a set $H_\omega^{(\lambda)}$ of $[H_\omega]$ such that $G_1^{(\lambda)}$ (consisting of the first α_1 and α_2 used in obtaining $H_\omega^{(\lambda)}$) and the boundary of $G_1^{(\lambda)}$ contain no point of N. Then, the corresponding region $H_1^{(\lambda)}$ contains no point of N. This follows by an argument previously used. Then, $[F]$ lies in the exterior of $H_1^{(\lambda)}$ and, therefore, in the exterior of $H_\omega^{(\lambda)}$. But $H_\omega^{(\lambda)}$ contains at least one point of $[F]$. This contradiction shows that $[F]$ cannot lie wholly in $S - H_\omega$. Let I denote $\overline{H}_\omega - J$. Then I_J is not vacuous. That I_J is unique and bounded follows as in Theorem 2 Theorems 3 – 10 can now be proved precisely as in the first part of this paper.

Theorem 11′ (identical with Theorem 11).

The proof of this theorem, based upon Axioms II, is precisely the proof given for Theorem 11.

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